

TECHNICAL NOTE

Estimating Holstein Lactation Curves with a Gamma Curve¹

D. WAYNE KELLOGG, N. SCOTT URQUHART,
and A. J. ORTEGA²

Departments of Dairy and Experimental Statistics
New Mexico State University
Las Cruces, NM 88003

ABSTRACT

Milk production records of healthy Holstein cows were analyzed with the model:

$$Y_i = \beta_1 t_i^{\beta_2} e^{-\beta_3 t_i} + \epsilon_i$$

where Y_i is milk production in kilograms at a specific time (t_i) in months. The parameters in this model were estimated with the techniques of intrinsically nonlinear regression. The milk production data came from four lactations of 36 cows and single lactations of 37 other cows which had completed at least one lactation. Production was collected monthly except that weekly means were used during the first 2 mo for the latter group. The first and third coefficients (β_1 and β_3) were lower for lactation 1 than for later lactations, i.e., estimated curves were lower and flatter for the first lactation than for later ones. Variances of deviations from the estimated curves were approximately equal after the 1st mo of lactation indicating that intrinsically nonlinear regression was the appropriate approach. Although coefficients varied for individual cows, most lactations were estimated closely by a gamma curve; coefficients of determination were close to 1.0 for means of cows but were not as consistent for individual cows. This technique allows statistical comparison of lactation curves of experimental groups of cows. This technique may be useful to extend incomplete lactation records, to compare daughters of sire groups, and to compare

production of cows receiving different nutritional or physiological treatments.

INTRODUCTION

The graph of daily milk production against time (usually for 305 days, 10 mo, for the dairy cow) is a lactation curve. The curve for a mature cow generally increases rapidly after parturition, reaches a peak after 30 to 90 days, then declines linearly until advanced pregnancy causes a sharper decline (2, 3, 4, 15, 18, 20). Several factors shape the curve of an individual cow causing considerable variation in peak production and slopes of the curve (1, 4, 7, 8, 9, 10, 12, 13, 15, 19).

Mathematical formulas have been proposed for inclining (3) and declining (2) phases of average lactation curves. In one instance, formulas for both aspects were combined to represent the entire curve (3). Methods which characterize lactation curves could allow statistical comparison of milk production for entire lactations and avoid restriction to the approximately linear phase. This would include the critical first months of lactation in nutritional or physiological experiments. An evaluation of sires could use parameters derived from lactations of daughters. Additionally, the shape of the curve might be incorporated into the process of extending lactations. A gamma-type curve was used to fit production of just a single cow and average production of progeny of eight bulls (20). Wood (20) linearized the model, implicitly assuming substantially more variable production at the peak than at extremes of the curve. Techniques of intrinsically nonlinear regression allow the assumption that a cow's production has equal variance throughout her 10-mo lactation. Our assumption does not conflict with common experience, as it first might seem. A group of cows usually does have higher variance of production around 2 mo than around 8 mo. Two factors other than random variation contribute to this comparison of variances. Cows

Received October 20, 1976.

¹ Journal article 610, Agricultural Experiment Station, New Mexico State University, Las Cruces 88003.

² Departamento de Recursos Pecuarios, Universidad Nacional de Colombia, Medellin.

TABLE 1. Estimation of four consecutive mean lactation curves from mean daily milk production of monthly observations of 36 Holsteins.

Month of lactation	Milk production							
	Lactation 1		Lactation 2		Lactation 3		Lactation 4	
	Observed	Estimated	Observed	Estimated	Observed	Estimated	Observed	Estimated
	(kg)							
1	22.0	22.1	29.5	29.5	31.4	31.4	31.3	31.6
2	23.2	22.9	29.9	29.7	32.3	32.1	33.1	32.4
3	22.7	22.6	27.7	27.9	29.9	30.2	30.5	30.5
4	21.7	21.9	25.0	25.5	27.1	27.4	26.9	27.7
5	20.7	21.1	23.0	22.9	24.5	24.5	24.6	24.7
6	20.0	20.1	20.5	20.3	22.0	21.6	21.7	21.8
7	19.3	19.1	18.4	18.0	19.1	18.9	19.5	19.0
8	18.4	18.1	16.3	15.8	17.1	16.5	17.4	16.5
9	17.4	17.2	13.6	13.8	14.1	14.3	14.0	14.3
10	15.9	16.2	11.4	12.0	11.7	12.3	11.8	12.3
Equation ^a	$\hat{Y} = 23.365t^{.154}e^{-.074t}$ R ² = .9878		$\hat{Y} = 34.716t^{.245}e^{-.162t}$ R ² = .9968		$\hat{Y} = 37.503t^{.286}e^{-.177t}$ R ² = .9976		$\hat{Y} = 37.838t^{.297}e^{-.181t}$ R ² = .9949	

^a $\hat{Y}$ = milk production in kilograms and t = time in months.

have different lactation curves so individuals following different curves will differ much more at the second than the eighth month. Secondly, the actual days postpartum for the second record of monthly production can range from about 35 to 70. Thus, early production records reflect much more diverse stages of lactation than do later records. By contrast our model accounts for both of these factors.

In this investigation, the parameters of a gamma curve were estimated with techniques of intrinsically nonlinear regression to fit lactation curves.

EXPERIMENTAL PROCEDURE

Milk production records of two groups of cows were used. Monthly Dairy Herd Improvement (DHI) records of the first four lactations of 36 Holstein cows provided data for one group. For single lactations of 37 other Holstein cows, daily production was averaged for each week during the initial 2 mo of lactation, and monthly DHI records were used for the remainder. Each cow of the latter group had completed at least one previous lactation. Only records from healthy cows were used; entire lactation records were discarded if at any point milk scored two or three on the California Mastitis Test or if milk production dropped dramatically because of illness.

During the first 2 mo of lactation, the latter group of cows was fed individually. Otherwise, all cows were housed and fed with groups of about 30 cows of similar production. Enough feed was offered to allow cows to consume about 5% more nutrients than required (16). Personnel, milking facilities, and milking pro-

cedures remained constant during data collection.

Daily milk production was regressed on the actual time after parturition when measurements were made by the model:

$$Y_i = \beta_1 t_i^{\beta_2} e^{-\beta_3 t_i} + \epsilon_i, \quad [1]$$

where Y_i is milk production in kilograms at a specific time (t_i) in months.

The parameters (β 's) were estimated with a locally developed computer program based on the principles of intrinsically nonlinear regression discussed by Draper and Smith (6). It utilized the computational algorithm given by Marquardt (14). Analyses of variance were used to distinguish among mean curves of different groups of cows. Maximum ratios of variance were used to determine significance of deviation from the estimated curves (17, Table 31). This statistical procedure provides the basis for comparing every variance with every other variance.

RESULTS AND DISCUSSION

The model [1] was fitted both to individual cow data and to means over 36 cows but within lactation. Table 1 presents observed mean data and fitted results for each lactation. The fit by the model was extremely close ($P < .005$) in each instance that means were used, as indicated by R^2 values (Table 1).

Means of each coefficient of gamma curves of 36 individual curves for each lactation appear in Table 2. Mean values vary from coefficients in Table 1 which were estimated from monthly means because we are dealing with a

TABLE 2. Mean estimated regression coefficients of individual gamma curves of 36 Holsteins for four lactations.

Lactation	First coefficient ^a		Second coefficient ^b		Third coefficient ^c	
	($\hat{\beta}_1$)	SE	($\hat{\beta}_2$)	SE	($\hat{\beta}_3$)	SE
1	23.7	.8	.119	.003	.061	.008
2	35.7	1.4	.250	.049	.160	.015
3	39.1	1.5	.324	.061	.186	.016
4	38.1	1.2	.395	.050	.206	.017

^aMultiplier, kg.

^bPower of time.

^cMultiplier in the exponent of e.

TABLE 3. Analyses of variance for estimated regression coefficients of individual gamma curves of four lactations of 36 Holsteins.

Source	df	First coefficient ^a	Second coefficient ^b	Third coefficient ^c
		(mean squares)		
Total	143			
Cows (C)	35	544**	.098	.012**
Lactations (L)	3	8,874***	.501	.153***
1 vs. 2, 3, 4	1	25,559***	1.123	.422***
2, 3, 4	2	532	.190	.019
Error (C × L)	105	176	.083	.006

Significant at $P < .01$.*Significant at $P < .005$.^aMultiplier.^bPower of time.^cMultiplier in the exponent of e .

nonlinear model. Coefficients in Table 2 were different ($P < .005$) across lactations (Table 3).

Because coefficients varied from cow to cow and from lactation to lactation, further analysis seemed necessary. The 36×4 estimates of each coefficient can be subjected to analysis of variance. The approximate validity of this procedure rests on the fact that estimates of each coefficient have approximately equal variance because production was evaluated at about the same time for all cows and lactations. Analysis of variance tests (Table 3) may be conservative because the error term contains cow by lactation interaction if it exists. From the mean squares the first and third coefficients change across cows and across lactations; the lactation differences occur because the curves of the first-calf heifers have a different shape from later lactations.

To interpret these results, how coefficients affect the curves needs to be considered. Because β_1 is a multiplier, it changes height of the curve across all times, but it has no impact on shape of the curves. From means of estimates of the first coefficient in Table 2 and the analysis of variance, the gamma curves fitted to the first lactation have much lower values than the curves for later lactations, a result consistent with other findings (1, 9, 10, 12, 13, 15). These gamma curves peak at β_2/β_3 , which lies between 1.5 and 2.0 mo for all lactations. The second coefficient (β_2) shapes the rising por-

tion of the curve while the third one (β_3) dominates how fast the curve drops off from its maximum. The curve for the first lactation remains fairly flat, compared with later ones, which become progressively steeper during this declining phase (Table 2).

The first and third coefficients (β_1 and β_2) did vary ($P < .01$) among individual cows (Table 3). Most individual disparity occurred during lactation 1 and was apparent in the range of distribution of R^2 (Table 4). Only four of 108 values of R^2 were below .8 for lactations 2, 3,

TABLE 4. Distribution of R^2 for gamma curve estimates of individual lactation curves of 36 Holsteins from 10 monthly observations.

Range of R^2 values	Lactation 1	Lactation 2	Lactation 3	Lactation 4
	(number of cows)			
>.90	9	30	30	25
.80 to .89	11	4	6	9
.70 to .79	4	1	0	1
.60 to .69	2	0	0	0
.50 to .59	2	0	0	0
.40 to .49	2	1	0	1
.30 to .39	3	0	0	0
.20 to .29	0	0	0	0
.10 to .19	1	0	0	0
<.10	2	0	0	0
Total	36	36	36	36

and 4. Sixteen of 36 values of R^2 were below .8 for lactation 1. Poor fit occurred when curves were relatively flat without distinct peaks in early lactation. Figures 1 and 2 illustrate curves of individual cows. Although peak production and slopes of curves varied, the model did fit well in each of these instances. Precision was greater when data were combined across cows than for most individual lactations.

Variability in lactation curves of cattle has been noted. Higher peak production usually has resulted in a more rapid decline (1, 9, 10, 12, 13, 15), and a gamma curve can display this property too. Interactions between parity and peak production have affected curves (1, 15); again gamma curves can behave this way.

Table 5 contains several summaries of the deviations of the observed productions from the gamma curves fitted to the individual cow data (denoted $Y - \hat{Y}$). Although the deviations in separate months within a lactation came from the same 36 cows, the homogeneity of the deviations across months reflects on the assumption of homogeneity of variance and the mean of the deviations on the fit of the model. Deviations were relatively small for each month, indicating that the fitted curves have little if any bias at any time. All possible pairwise comparisons of variance were made by the technique described at the end of the proced-

ures section. The variance in the 1st mo was lower than the variance in some later month for all lactations; otherwise, no differences among variances were observed. This strongly supports the assumption of homogeneity of variance for months 2 through 10. Variance really may be slightly lower during the 1st mo, but we doubt this. The data do not reflect lower variance early in the lactation. The observed low variance of the residuals could result from the estimation procedure because it seemed to need a good fit on the first point to get a reasonable fit at succeeding points.

These considerations indicate that the scatter of data around the estimated lactation curves was relatively uniform and tends to substantiate use of the intrinsically nonlinear model in this investigation. Wood (20, 21) regressed the logarithm of Y on X and the logarithm of X . This process implies a model which involves the parameters (β 's) in the same way as ours but involves the sampling deviations or "errors" (ϵ_i 's) differently. Because quantities which add on a logarithmic scale multiply on the data scale, his model effectively assumed that the sampling deviations entered his model in a multiplicative manner. Consequently the use of his model assumed greater variation of production about the peak of the curve than at the extremes. In this study, variances were not

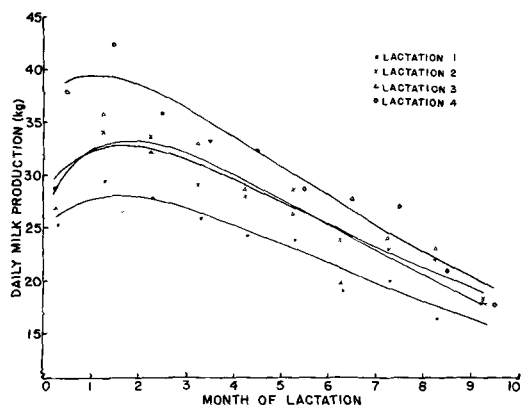


FIG. 1. Estimated lactation curves of cow 619 representing equations: $\hat{Y} = 30.314t^{.248}e^{-.121t}$ for lactation 1 ($R^2 = .9187$); $\hat{Y} = 35.088t^{.265}e^{-.124t}$ for lactation 2 ($R^2 = .9314$); $\hat{Y} = 35.062t^{.350}e^{-.149t}$ for lactation 3 ($R^2 = .8225$); $\hat{Y} = 44.181t^{.202}e^{-.128t}$ for lactation 4 ($R^2 = .9389$), where $\hat{Y}$ = milk production in kilograms and t = time in months.

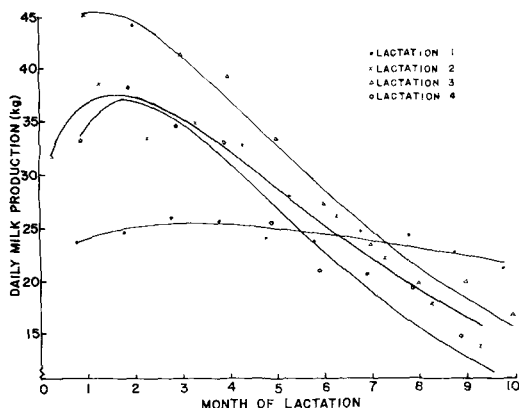


FIG. 2. Estimated lactation curves of cow 627 representing equations: $\hat{Y} = 24.306t^{.186}e^{-.053t}$ for lactation 1 ($R^2 = .7714$); $\hat{Y} = 41.593t^{.431}e^{-.200t}$ for lactation 2 ($R^2 = .9573$); $\hat{Y} = 52.696t^{.355}e^{-.197t}$ for lactation 3 ($R^2 = .9857$); $\hat{Y} = 40.370t^{.672}e^{-.282t}$ for lactation 4 ($R^2 = .9543$), where $\hat{Y}$ = milk production in kilograms and t = time in months.

TABLE 5. Estimated means, standard errors of means, and variances of deviations ($Y - \hat{Y}$) from regression curves of individual lactations of 36 Holsteins for four lactations.

Lactation	Month									
	1	2	3	4	5	6	7	8	9	10
	(kg)									
Lactation 1										
Mean	-.22	.48	.07	-.35	-.20	-.05	.18	-.01	.35	-.25
SE	.11	.21	.16	.19	.20	.22	.19	.20	.16	.19
Variance	.93	3.57*	2.16	2.72	3.35*	3.94*	2.88	3.18*	2.04	2.85
Lactation 2										
Mean	-.04	.27	-.09	-.42	-.05	-.01	.32	.68	-.09	-.68
SE	.15	.30	.29	.26	.32	.24	.26	.22	.18	.29
Variance	1.69	7.30*	6.50*	5.53	8.20*	4.49	5.37	3.81	2.62	6.40*
Lactation 3										
Mean	.01	.37	-.44	-.49	.05	.71	.12	.77	-.36	-1.12
SE	.20	.34	.31	.36	.33	.34	.28	.27	.31	.33
Variance	3.23	9.31	7.54	10.53	8.58	9.20	6.09	5.81	7.62	8.70
Lactation 4										
Mean	-.20	.65	-.05	-.85	-.10	.03	.54	.90	-.35	-.69
SE	.21	.45	.34	.34	.32	.27	.30	.33	.26	.30
Variance	3.66	16.00*	9.09	9.52	7.93	5.73	7.19	8.38	5.33	7.35

*Variance of these months greater ($P < .05$) than variance of month 1 within lactation.

lower ($P > .05$) during the latter months than they were in mo 1 to 4.

Data from DHI provided limited information during early lactation when production ordinarily increases rapidly. Table 6 contains means of 37 lactation curves; during early lactation daily production was averaged to obtain weekly means. The gamma curve fit closely ($R^2 = .9978$), again indicating the precision with which the model fits lactation curves. While precision of estimation was high both here and with DHI data, increasing the number of weighings provided more accurate measurement of production in early lactation. In this instance, $\hat{\beta}_1$ was higher than mean values of $\hat{\beta}_1$ in Table 2. This reflects higher peak production of these cows compared with those presented earlier. Data in Table 6 were gained later, through the use of different cows which were fed and housed individually during early lactation. Peak production in Table 6 occurred earlier than in

Table 2. This may be due to more data available during early lactation. Both $\hat{\beta}_2$ and $\hat{\beta}_3$ were within ranges given in Table 2 for different lactations of other cows.

The approximately linear phase of the lactation curve has been used extensively as the period of nutritional experimentation with cows (5, 11). These methods have restricted length of experiments and have not been applicable to the initial 2 mo of lactation. The gamma curve possibly could allow comparison of entire lactations in many experiments. Excellent techniques available for extending incomplete lactations might shorten the time required for experiments during the initial phase of lactation.

Parameters of the gamma curve have seen limited use in evaluation of sires (20). Curves of daughters of sires were fitted, then statistics were compared.

An additional application may exist in extending partial or incomplete lactations. Consideration of shape of curves during early months of lactation should indicate persistency of milk yield. Incorporation of parameters of the gamma curve into current processes of extending lactations needs study.

In conclusion, intrinsically nonlinear techniques in this investigation closely fitted lactation curves of individual cows unless curves were relatively flat. In every instance, curves of means of 36 or 37 cows were fitted precisely. These intrinsically nonlinear techniques provide a method to compare lactation curves of different groups of cows.

ACKNOWLEDGMENT

The authors express thanks to Morris D. Finkner and Gerald S. Rogers for reviewing the manuscript.

REFERENCES

- 1 Appleman, R. D., S. D. Musgrave, and R. D. Morrison. 1969. Extending incomplete lactation records of Holstein cows with varying levels of production. *J. Dairy Sci.* 52:360.
- 2 Brody, S., A. C. Ragsdale, and C. W. Turner. 1923. The rate of decline of milk secretion with the advance of the period of lactation. *J. Gen. Physiol.* 5:441.
- 3 Brody, S., C. W. Turner, and A. C. Ragsdale. 1924. The relation between the initial rise and the subsequent decline of milk secretion following parturition. *J. Gen. Physiol.* 6:541.

TABLE 6. Estimation of mean lactation curves of 37 Holsteins from weekly mean milk production for 2 mo, then monthly observations for 8 mo.

Mean time postpartum (days)	Milk production	
	Observed	Estimated ^a
	(kg)	
5	26.8	26.7
12	30.9	31.5
19	34.0	33.6
26	35.1	34.6
33	35.0	35.0
40	34.9	35.0
47	34.7	34.7
54	34.1	34.3
67	33.1	33.1
91	30.1	30.3
116	27.1	27.2
149	23.6	23.3
177	19.8	20.2
209	17.1	16.9
241	15.2	14.2
270	11.9	12.1
298	9.5	10.3

^aEstimated by the equation, $\hat{Y} = 42.538t^{-.235}e^{-.197t}$, where $\hat{Y}$ is milk production in kilograms and t is time in mo. $R^2 = .9978$ and is significant ($P < .005$).

- 4 Campbell, J. R., and R. T. Marshall. 1975. *The Science of Providing Milk for Man*. McGraw Hill Book Co., New York.
- 5 Cunningham, P. J., and F. G. Owen. 1971. Statistical methods for improving sensitivity in dairy cattle feeding experiments. *J. Dairy Sci.* 54:503.
- 6 Draper, N. R., and H. Smith. 1966. *Applied Regression Analysis*. John Wiley and Sons, Inc., New York.
- 7 Freeman, A. E. 1973. Age adjustment of production records: History and basic problems. *J. Dairy Sci.* 56:941.
- 8 Fritz, G. R., L. D. McGilliard, and D. E. Madden. 1960. Environmental influences on regression factors for estimating 305-day production from part lactations. *J. Dairy Sci.* 43:1108.
- 9 Gooch, M. 1935. An analysis of the time change in milk production in individual lactations. *J. Agr. Sci.* 25:71.
- 10 Lamb, R. C., and L. D. McGilliard. 1960. Variables affecting ratio factors for estimating 305-day production from part lactation. *J. Dairy Sci.* 43:519.
- 11 Lucas, H. L. 1960. Critical features of good dairy feeding experiments. *J. Dairy Sci.* 43:193.
- 12 Madden, D. E., J. L. Lush, and L. D. McGilliard. 1955. Relations between parts of lactations and producing ability of Holstein cows. *J. Dairy Sci.* 38:1264.
- 13 Mahadevan, P. 1951. The effects of environment and heredity on lactation. II. Persistency of lactation. *J. Agr. Sci.* 41:89.
- 14 Marquardt, D. W. 1970. Generalized inverses, ridge regression, biased linear estimation, and nonlinear estimation. *Technometrics* 12:591.
- 15 Miller, D. D. 1975. Lactation and other production curves as criteria for culling dairy cows. *New Mexico Agr. Exp. Bull.* 632.
- 16 National Research Council. 1971. *Nutritional requirements of dairy cattle*. National Academy of Sciences. Washington, DC.
- 17 Pearson, E. S., and H. O. Hartley. 1958. *Biometrika Tables for Statisticians*, Vol. 1, Cambridge Univ. Press, London.
- 18 Turner, C. W. 1926. A quantitative form of expressing persistency of milk or fat secretion. *J. Dairy Sci.* 9:203.
- 19 Van Vleck, L. D., and C. R. Henderson. 1961. Ratio factors for adjusting monthly test-day data for age and season of calving and ratio factors for extending part lactation records. *J. Dairy Sci.* 44:1093.
- 20 Wood, P. D. P. 1967. Algebraic model of the lactation curve in cattle. *Nature* 216:164.
- 21 Wood, P. D. P. 1970. A note on the repeatability of parameters of the lactation curve in cattle. *Anim. Prod.* 12:535.